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Simulation and estimation for the fractional Yule process

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Abstract In this paper, we propose some representations of a generalized linear birth process called fractional Yule process (fYp). We also derive the probability distributions of the random birth and sojourn times. The inter-birth time distribution and the representations then yield algorithms on how to simulate sample paths of the fYp. We also attempt to estimate the model parameters in order for the fYp to be usable in practice. The estimation procedure is then tested using simulated data as well. We also illustrate some major characteristics of fYp which will be helpful for real applications.

Keywords Yule–Furry process · fractional calculus · Mittag–Leffler · Wright · Poisson process · birth process

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1 Introduction

The pure birth process is undoubtedly considered as one of the simplest branching processes. It has a Markovian structure and has already been extensively studied in the past. When the birth rate is linear, it is then usually called

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the pure linear birth or classical Yule or Yule–Furry process (Yp). The pure linear birth process has been introduced by McKendrick (1914), and has been widely used to model various stochastic dynamical systems such as cosmic showers, epidemics, and population growth to name a few.

For the sake of completeness, we review some known properties of the classical Yule process which will be used in the succeeding discussion. Let $\mathfrak{N}(t)$ be the number of individuals in a Yule process with a single initial progenitor and birth intensity $\lambda > 0$. The kth state probability or the probability of having exactly k individuals $\mathfrak{p}_k(t) = \Pr\{\mathfrak{N}(t) = k \mid \mathfrak{N}(0) = 1\}$ in a growing population at time $t > 0$ solves the following Cauchy problem:

$$\begin{cases} \frac{d}{dt}\mathfrak{p}_k(t) = -\lambda k \mathfrak{p}_k(t) + \lambda(k-1)\mathfrak{p}_{k-1}(t), & k \geq 1, \\ \mathfrak{p}_k(0) = \begin{cases} 1, & k = 1, \\ 0, & k > 1, \end{cases} \end{cases} \tag{1.1}$$

where $\mathfrak{p}_0(0) = 0$. The explicit solution to (1.1) is

$$\mathfrak{p}_k(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}, \qquad t > 0,\ k \geq 1,$$

with mean $\mathbf{E}\mathfrak{N}(t) = e^{\lambda t}$. To make the Yule process more flexible in taking into account more complex non-Markovian behaviour, some authors (Uchaikin et al. (2008), Orsingher and Polito (2010)) proposed a more general model called the fractional Yule process (fYp). A similar generalization of other point processes such as the Poisson process has previously been carried out by Repin and Saichev (2000), Jumarie (2001), Laskin (2003), Wang and Wen (2003), Mainardi et al. (2004), Wang et al. (2006), Wang et al. (2007), Mainardi et al. (2005), Cahoy (2007), Uchaikin and Sibatov (2008), Uchaikin et al. (2008) and Beghin and Orsingher (2009).

The aim of this paper is twofold: We want to derive related representations of fYp in terms of some classical or standard processes, and we want to construct algorithms on how to simulate a fYp and estimate the parameters.

We organize the rest of the paper as follows: In Section 2, we show the fractional generalization of the pure linear birth process. In Section 3, it is illustrated that a pure linear birth process can also be viewed as a classical linear pure birth process with Wright-distributed random rates evaluated on a stretched time scale, i.e.,

$$\mathfrak{N}^\nu(t) \stackrel{d}{=} \mathfrak{N}_{\varXi}(t^\nu), \qquad \nu \in (0,1],$$

where $\varXi$ is a random variable having the Wright probability density function

$$W_{-\nu,1-\nu}(-\xi) = \sum_{r=0}^{\infty} \frac{(-\xi)^r}{r!\varGamma(1-\nu(r+1))}. \tag{1.2}$$

Furthermore, some Poisson-related representations are proved. In Section 4, we derive the birth and inter-birth time distributions. The structural representation, fractional moments of the sojourn and birth times are also shown. In

Section 5, we generate sample paths of a fYp using our algorithms. In Section 6, an estimation procedure is proposed using the moments of the log-transformed data, and some empirical results are showed as well. Section 7 concludes the paper with a discussion on the key points and possible extensions of this study.

2 Generalization of the Yule process

The fractional generalization of the Cauchy problem (1.1) was first carried out in Uchaikin et al. (2008), Section 8, and is described as follows: The authors defined the following difference-differential equations governing the state probabilities $p_k^\nu(t) = \Pr\{N^\nu(t) = k \mid N^\nu(0) = 1\}$:

$$\frac{\mathfrak{d}^\nu}{\mathfrak{d}t^\nu} p_k^\nu(t) = \lambda \left[\sum_{l=1}^{k-1} p_l^\nu(t) p_{k-l}^\nu(t) - p_k^\nu(t) \right] + \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta_{k,1}, \qquad \nu \in (0,1],\ k \geq 1, \tag{2.1}$$

where the initial condition

$$p_k^\nu(0) = \begin{cases} 1, & k = 1, \\ 0, & k > 1, \end{cases}$$

is incorporated into equation (2.1) through the Kronecker delta $\delta_{k,1}$. The fractional derivative appearing in (2.1) is the so-called Riemann–Liouville operator, and is defined as

$$\begin{cases} \frac{\mathfrak{d}^\nu}{\mathfrak{d}t^\nu} f(t) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\nu} ds, & \nu \in (0,1), \\ f'(t), & \nu = 1. \end{cases} \tag{2.2}$$

Furthermore, the mean number of individuals in the system was found to be

$$\mathbf{E}\big[N^\nu(t)\big] = E_{\nu,1}(\lambda t^\nu), \qquad t > 0,\ \nu \in (0,1], \tag{2.3}$$

where

$$E_{\alpha,\beta}(\tau) = \sum_{r=0}^{\infty} \frac{\tau^r}{\Gamma(\alpha r + \beta)}, \qquad \alpha, \beta, \in \mathbb{R}^+,\ \tau \in \mathbb{R},$$

is the generalized Mittag–Leffler function.

Let $\mathfrak{N}^\nu(t)$ be the number of individuals in a fractional linear birth process or fractional Yule or Yule–Furry process (fYp) up to the time $t > 0$. The state probabilities $\mathfrak{p}_k^\nu(t) = \Pr\{\mathfrak{N}^\nu(t) = k \mid \mathfrak{N}^\nu(0) = 1\}$ solve the following Cauchy problem:

$$\begin{cases} \frac{d^\nu}{dt^\nu} \mathfrak{p}_k^\nu(t) = -\lambda k \mathfrak{p}_k^\nu(t) + \lambda(k-1)\mathfrak{p}_{k-1}^\nu(t), & k \geq 1, \\ \mathfrak{p}_k^\nu(0) = \begin{cases} 1, & k = 1, \\ 0, & k > 1, \end{cases} \end{cases} \tag{2.4}$$

which is also a fractional generalization of (1.1). The fractional derivative involved in (2.4) is now the Caputo operator, and is defined as

$$\begin{cases} \frac{d^\nu}{dt^\nu} f(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f'(s)}{(t-s)^\nu} ds, & \nu \in (0,1), \\ f'(t), & \nu = 1. \end{cases} \tag{2.5}$$

Moreover, the Riemann–Liouville (2.2) and the Caputo (2.5) fractional derivatives are linked together by the following relation (see Kilbas et al. (2006), page 91):

$$\frac{d^\nu}{dt^\nu} f(t) = \frac{\mathfrak{d}^\nu}{\mathfrak{d}t^\nu} f(t) - \frac{f(0)}{\Gamma(1-\nu)} t^{-\nu}, \qquad \nu \in (0,1). \tag{2.6}$$

From (2.6), it is easy to see that both fractional derivatives coincide when $f(0) = 0$ for each $k > 1$. The solution to the Cauchy problem (2.4) is

$$\mathfrak{p}_k^\nu(t) = \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu), \qquad k \geq 1, \; \nu \in (0,1]. \tag{2.7}$$

Note that the mean number of individuals $\mathbf{E}\left[\mathfrak{N}^\nu(t)\right]$ in the fractional Yule process is the same as (2.3), and the variance can be calculated as

$$\mathbf{V}ar\left(\mathfrak{N}^\nu(t)\right) = 2E_{\nu,1}\left(2\lambda t^\nu\right) - E_{\nu,1}\left(\lambda t^\nu\right) - E_{\nu,1}^2\left(\lambda t^\nu\right)$$

From here on, we emphasize that the fractional derivative operation is performed in Caputo's sense.

3 Stretched Yule process with random rates and related representations

In this section, we present some relevant and interesting representations of the fractional Yule process (fYp). We start by proving a subordination relation that links the fractional Yule process with its classical counterpart.

Theorem 3.1 *Let $\mathfrak{N}^\nu(t)$ be the number of individuals in a fractional Yule process at time $t > 0$. Then the following equality in distribution holds:*

$$\mathfrak{N}^\nu(t) \stackrel{d}{=} \mathfrak{N}(T_{2\nu}(t)), \tag{3.1}$$

where $\mathfrak{N}(t)$ is a classical Yule process, $\nu \in (0,1]$, and $T_{2\nu}(t)$ is a random time whose distribution coincides with the solution to the following Cauchy problem

$$\begin{cases} \frac{\partial^{2\nu}}{\partial t^{2\nu}} g(x,t) = \frac{\partial^2}{\partial x^2} g(x,t), & x > 0, \\ \frac{\partial}{\partial x} g(x,t)\big|_{x=0} = 0, \\ g(x,0) = \delta(x), \end{cases} \tag{3.2}$$

with the initial condition $g_t(x,0) = 0$, when $1/2 < \nu \leq 1$.

Proof Let $G^\nu(u,t)$, $t>0$, $|u|<1$, be the probability generating function of the fractional Yule process. To prove (3.1)*, it is sufficient to observe that*

$$\begin{aligned}
&\int_0^\infty e^{-zt} G^\nu(u,t)dt \\
&= \int_0^\infty e^{-zt} \sum_{k=1}^\infty u^k \mathfrak{p}_k^\nu(t)dt \\
&= \int_0^\infty e^{-zt} \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu)dt \\
&= \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} \frac{z^{\nu-1}}{z^\nu + \lambda l} \\
&= \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} z^{\nu-1} \int_0^\infty e^{-s(\lambda l + z^\nu)} ds \\
&= \int_0^\infty \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-s\lambda l} z^{\nu-1} e^{-sz^\nu} ds \\
&= \int_0^\infty \sum_{k=1}^\infty u^k \sum_{l=1}^k \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l s} \int_0^\infty e^{-zt} Pr\{T_{2\nu}(t) \in ds\}\, dt \\
&= \int_0^\infty e^{-zt} \left[\sum_{k=1}^\infty u^k \int_0^\infty Pr\{\mathfrak{N}(s) = k\}\, Pr\{T_{2\nu}(t) \in ds\} \right] dt \\
&= \int_0^\infty e^{-zt} \left[\sum_{k=1}^\infty u^k Pr\{\mathfrak{N}(T_{2\nu}(t)) = k\} \right] dt. \quad \square
\end{aligned}$$

Remark 3.1 *Note that, the solution to* (3.2)*, also solves the fractional differential equation*

$$\frac{\partial^\nu}{\partial t^\nu} g(x,t) = -\frac{\partial}{\partial x} g(x,t). \tag{3.3}$$

Remark 3.2 *In the proof of Theorem 3.1, we used the Laplace transform of $Pr\{T_{2\nu}(t) \in ds\}$ which is*

$$\int_0^\infty e^{-zt} Pr\{T_{2\nu}(t) \in ds\} = z^{\nu-1} e^{-sz^\nu} ds, \qquad s>0.$$

In the next Theorem, we derive a random-rate representation of the fractional Yule process using the preceding subordination relation.

Theorem 3.2 *(Representation A) Let $t>0$ and $\nu \in (0,1]$. Then the following equality in distribution holds:*

$$\mathfrak{N}^\nu(t) \overset{d}{=} \mathfrak{N}_\Xi(t^\nu), \tag{3.4}$$

where $\mathfrak{N}_{\Xi}(t^\nu)$ *is a classical linear birth process with random rate* $\lambda\Xi$ *evaluated at* t^ν, Ξ *is a Wright-distributed random variable with probability density function* $W_{-\nu,1-\nu}(-\xi)$ *in (1.2).*

Proof *To prove equality* (3.4)*, we use the subordination relation* (3.1) *as follows:*

$$\begin{aligned}
&Pr\{\mathfrak{N}^\nu(t) = k \mid \mathfrak{N}^\nu(0) = 1\} \qquad (3.5)\\
&= \int_0^\infty Pr\{\mathfrak{N}(s) = k \mid \mathfrak{N}(0) = 1\}\, Pr\{T_{2\nu}(t) \in ds\}\\
&= \int_0^\infty \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l s} t^{-\nu} W_{-\nu,1-\nu}(-t^{-\nu}s)ds\\
&= \int_0^\infty \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l \xi t^\nu} W_{-\nu,1-\nu}(-\xi)d\xi\\
&= \int_0^\infty Pr\{\mathfrak{N}_\xi(t^\nu) = k \mid \mathfrak{N}_\xi(0) = 1\}\, W_{-\nu,1-\nu}(-\xi)d\xi,
\end{aligned}$$

and this leads to (3.4)*.* □

Note that in the second step of formula (3.5), we used the explicit form of the solution to the fractional diffusion equation (3.2) which is (see Podlubny (1999), formula (4.22), page 142)

$$\Pr\{T_{2\nu}(t) \in ds\} = t^{-\nu} W_{-\nu,1-\nu}(-t^{-\nu}s)ds, \qquad s > 0.$$

Remark 3.3 *As noted above, representation* (3.4) *holds for the one-dimensional state probability distribution* $\mathfrak{p}_k^\nu(t)$, $t > 0$, $k \geq 1$. *This, however is sufficient in the sense that the process* $\mathfrak{N}_\Xi(t^\nu)$ *has distribution that solves* (2.4)*.*

We now prove a further interesting representation of the fractional Yule process in terms of a specific mixed non-homogeneous Poisson process.

Starting from the second-to-last step of formula (3.5), we obtain

$$\begin{aligned}
\mathfrak{p}_k^\nu(t) &= \int_0^\infty \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} e^{-\lambda l \xi t^\nu} W_{-\nu,1-\nu}(-\xi)d\xi\\
&= \int_0^\infty e^{-\lambda \xi t^\nu} \left[1 - e^{-\lambda \xi t^\nu}\right]^{k-1} W_{-\nu,1-\nu}(-\xi)d\xi\\
&= \int_0^\infty \frac{1}{\left[e^{\lambda \xi t^\nu}\right]^k} \left[e^{\lambda \xi t^\nu} - 1\right]^{k-1} W_{-\nu,1-\nu}(-\xi)d\xi.
\end{aligned}$$

Recalling the identity

$$\int_0^\infty e^{-ax} x^r dx = a^{-(r+1)} r!, \qquad r \in \mathbb{N},\ \Re(a) > 0,$$

we get

$$\mathfrak{p}_k^\nu(t) = \int_0^\infty \int_0^\infty e^{-\omega e^{\lambda \xi t^\nu}} \omega^{k-1} \frac{\left[e^{\lambda\xi t^\nu}-1\right]^{k-1}}{(k-1)!} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi \tag{3.6}$$
$$= \int_0^\infty \int_0^\infty \frac{e^{-\omega\left[e^{\lambda\xi t^\nu}-1\right]}\omega^{k-1}\left[e^{\lambda\xi t^\nu}-1\right]^{k-1}}{(k-1)!} e^{-\omega} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi$$
$$= \int_0^\infty \int_0^\infty \frac{e^{-\int_0^{t^\nu}\omega\lambda\xi e^{\lambda\xi s}ds}\left[\int_0^{t^\nu}\omega\lambda\xi e^{\lambda\xi s}ds\right]^{k-1}}{(k-1)!} e^{-\omega} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi.$$

Thus, we have obtained a representation in terms of a mixed non-homogeneous Poisson process with intensity function

$$\lambda(t) = \Omega\lambda\Xi e^{\lambda\Xi t}, \qquad t>0,$$

where the distribution of Ω is negative-exponential with mean equal to 1, and Ξ has probability density function (1.2). Note that the random variable Ω, conditional on $\Xi=\xi$, is such that

$$\frac{N_\xi(t^\nu)}{\mathbf{E}N_\xi(t^\nu)} \xrightarrow{\text{a.s.}} \Omega,$$

as $t\to\infty$ (see e.g. Keiding (1974), Waugh (1970), Harris (2002)).

Remark 3.4 *A simple change of variable also allows us to obtain a representation in terms of a mixed non-homogeneous Poisson process evaluated at the random time $T_{2\nu}(t)$, $t>0$. From the second step of formula* (3.6)*, we have*

$$\mathfrak{p}_k^\nu(t) = \int_0^\infty \int_0^\infty \frac{e^{-\omega\left[e^{\lambda\xi t^\nu}-1\right]}\omega^{k-1}\left[e^{\lambda\xi t^\nu}-1\right]^{k-1}}{(k-1)!} e^{-\omega} W_{-\nu,1-\nu}(-\xi) d\omega\, d\xi$$
$$= \int_0^\infty \int_0^\infty \frac{e^{-\omega\left[e^{\lambda s}-1\right]}\omega^{k-1}\left[e^{\lambda s}-1\right]^{k-1}}{(k-1)!} e^{-\omega} \frac{1}{t^\nu} W_{-\nu,1-\nu}\left(-\frac{s}{t^\nu}\right) ds\, d\omega.$$

Consider a non-homogeneous Poisson process $\mathrm{N}(t)$ *with intensity function* $\lambda(t)=\Omega\lambda e^{\lambda t}$. *Then the state probabilities of the fractional Yule process can be written as*

$$\mathfrak{p}_k^\nu(t) = \int_0^\infty e^{-\omega} \int_0^\infty Pr\{\mathrm{N}(s)=k-1\}\, Pr\{T_{2\nu}(t)\in ds\}\, d\omega \tag{3.7}$$
$$= \mathbf{E}_\Omega \mathrm{N}(T_{2\nu}(t)).$$

In addition, the subordinated non-homogeneous Poisson process $\mathrm{N}(T_{2\nu}(t))$ conditioned on $\Omega=\omega$ could be interesting as the fractional homogeneous Poisson process admits a similar representation (Beghin and Orsingher, 2009).

Let $q_k^\nu(t)$ be the state probabilities of $\mathrm{N}(T_{2\nu}(t))$, i.e.,

$$q_k^\nu(t) = \Pr\left\{\mathrm{N}(T_{2\nu}(t))=k-1\right\}, \qquad t>0,\ k\geq 1.$$

Then

$$q_k^\nu(t) = \int_0^\infty \frac{e^{-\omega\left[e^{\lambda s}-1\right]}\omega^{k-1}\left[e^{\lambda s}-1\right]^{k-1}}{(k-1)!} \Pr\left\{T_{2\nu}(t) \in ds\right\}. \tag{3.8}$$

Applying the Laplace transform to (3.8), we have

$$\begin{aligned}\int_0^\infty e^{-zt} q_k^\nu(t) dt &= \int_0^\infty \frac{e^{-\omega\left[e^{\lambda s}-1\right]}\omega^{k-1}\left[e^{\lambda s}-1\right]^{k-1}}{(k-1)!} z^{\nu-1} e^{-sz^\nu} ds \\ &= \int_0^\infty e^\omega \frac{e^{-\omega e^{\lambda s}}\omega^{k-1}\left[1-e^{\lambda s}\right]^{k-1}}{(k-1)!} (-1)^{k-1} z^{\nu-1} e^{-sz^\nu} ds,\end{aligned}$$

and by taking into account the relations

$$e^{-\omega e^{\lambda s}} = \sum_{l=0}^\infty \frac{(-\omega)^l e^{\lambda s l}}{l!},$$

$$\left[1-e^{\lambda s}\right]^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{\lambda s j},$$

we arrive at the equality

$$\begin{aligned}&\int_0^\infty e^{-zt} q_k^\nu(t) dt \qquad (3.9)\\ &= \int_0^\infty \frac{e^\omega}{(k-1)!}(-1)^{k-1}\omega^{k-1} \sum_{l=0}^\infty \sum_{j=0}^{k-1} \frac{(-\omega)^l}{l!} e^{\lambda l s} \binom{k-1}{j} (-1)^j e^{\lambda s j} z^{\nu-1} e^{-sz^\nu} ds \\ &= \frac{e^\omega}{(k-1)!}(-1)^{k-1}\omega^{k-1} \sum_{l=0}^\infty \frac{(-\omega)^l}{l!} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j z^{\nu-1} \int_0^\infty e^{-s[z^\nu-\lambda(l+j)]} ds \\ &= \frac{e^\omega}{(k-1)!}(-\omega)^{k-1} \sum_{l=0}^\infty \frac{(-\omega)^l}{l!} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{z^{\nu-1}}{z^\nu - \lambda(l+j)}.\end{aligned}$$

Applying the inverse Laplace transform to equation (3.9), we obtain the explicit expression of the state probabilities as

$$q_k^\nu(t) = \frac{e^\omega(-\omega)^{k-1}}{(k-1)!} \sum_{l=0}^\infty \frac{(-\omega)^l}{l!} \sum_{j=1}^{k} \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}\left[\lambda(l+j)t^\nu\right], \quad k \geq 1. \tag{3.10}$$

Remark 3.5 *From equation* (3.10)*, it is straightforward to obtain the classical form of the state probabilities of the (conditional) non-homogeneous Poisson process (*$\nu = 1$*) with intensity function* $\lambda(t) = \omega\lambda e^{\lambda t}, t > 0$*.*

We introduce a definition and a lemma below which will be helpful in transforming fYp into a non-homogeneous Poisson process with rate 1. In order to do so, we present here the standard definition, by means of a Mellin–Barnes type integral, of the so-called Fox function:

$$H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_p,B_p)\end{matrix}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Theta(z)x^{-z}dz,\quad x\neq 0,\quad (3.11)$$

where

$$\Theta(z)=\frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+B_jz)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-A_jz)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_jz)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+A_jz)\right\}}.\quad (3.12)$$

Each empty product is interpreted as unity. For more information on Fox functions we refer to Mathai et al. (2010).

Definition 3.1 *Let $\mathfrak{T}^\nu(t)$ be a random time process whose one-dimensional distribution is given by*

$$Pr\{\mathfrak{T}^\nu(t)\in ds\}=h(t,s)ds=t^{-\frac{1}{\nu}}H_{1,1}^{1,0}\left[t^{-\frac{1}{\nu}}s\,\middle|\,\begin{matrix}(1-1/\nu,1/\nu)\\(0,1)\end{matrix}\right]ds,$$

where $t>0$, $s>0$, $\nu\in(0,1]$. Furthermore, $h(t,s)$ has Mellin transform

$$\int_0^\infty s^{\eta-1}h(t,s)ds=\frac{\Gamma(\eta)}{\Gamma\left(1-\frac{1}{\nu}+\frac{1}{\nu}\eta\right)}t^{\frac{\eta-1}{\nu}}.\quad (3.13)$$

Lemma 3.1 *Let $\mathfrak{N}^\nu(t)$ be a fractional Yule process with rate $\lambda>0$ and $t>0$. Then the process $\mathfrak{N}^\nu(\mathfrak{T}^\nu(t))$ is a classical Yule process with rate λ.*

Proof Define $G^\nu(u,t)$ and $G(u,t)$, $t>0$, $|u|\le 1$ as the probability generating functions of fYp and the classical Yule process, respectively. Then

$$\int_0^\infty G^\nu(u,s)h(t,s)ds=\int_0^\infty\sum_{k=1}^\infty u^k\sum_{j=1}^k\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda j s^\nu)h(t,s)ds.$$

In the following we use the Mellin–Barnes representation of the Mittag–Leffler function

$$E_{\nu,1}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)}(-x)^{-z}dz,\qquad \nu>0,\ x\neq 0$$

(see Kilbas et al. (2006), page 41, formula (1.8.14)). Note that when $\nu=1$ we retrieve the Mellin–Barnes representation of the exponential function

$$e^x=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(z)(-x)^{-z}dz,\qquad x\neq 0.\quad (3.14)$$

(see Paris and Kaminski (2001), page 89, formula (3.3.2)).

We obtain

$$
\begin{aligned}
&\int_0^\infty G^\nu(u,s)h(t,s)ds \\
&= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} \frac{(-1)^{j-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)} (\lambda j)^{-z} \int_0^\infty \frac{h(t,s)}{s^{\nu z}} ds\, dz.
\end{aligned}
$$

Applying formula (3.13)*, we can write*

$$
\begin{aligned}
&\int_0^\infty G^\nu(u,s)h(t,s)ds \qquad (3.15)\\
&= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} \frac{(-1)^{j-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1-\nu z)} (\lambda j)^{-z} \frac{\Gamma(1-\nu z)}{\Gamma(1-z)} t^{-z} dz \\
&= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} \frac{(-1)^{j-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)(\lambda j t)^{-z} dz \\
&= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} e^{-\lambda j t} \\
&= \sum_{k=1}^\infty u^k e^{-\lambda t} \left[1-e^{-\lambda t}\right]^{k-1} \\
&= G(u,t). \qquad \square
\end{aligned}
$$

Remark 3.6 *Note that it is straightforward to generalize Lemma 3.1 to the more general (non-linear) case.*

Remark 3.7 *Letting* $u=1$ *in* (3.15)*, we have*

$$
\begin{aligned}
&\sum_{k=1}^\infty \int_0^\infty \mathfrak{p}_k^\nu(s)h(t,s)ds = \sum_{k=1}^\infty \mathfrak{p}_k(t) \\
&\Leftrightarrow \int_0^\infty h(t,s)ds = 1.
\end{aligned}
$$

Theorem 3.3 *Consider a fractional Yule process* $\mathfrak{N}^\nu(t)$ *with birth rate* $\lambda > 0$*,* $t > 0$*, and* $\nu \in (0,1]$*. Then the random time-changed process*

$$
\mathfrak{N}^\nu \left[\mathfrak{T}^\nu \left(\frac{1}{\lambda} \log \left(\frac{t}{\Omega} + 1 \right) \right) \right]
$$

has one-dimensional distribution which coincides with that of a non-homogeneous Poisson process $M(t)$ *with rate 1.*

Proof It readily follows from (3.7)*, Lemma 3.1 and Theorem 1 of Kendall (1966).*

4 Wait and sojourn time distributions

We now show that the sojourn or inter-birth time of fYp follows the Mittag–Leffler distribution. Let T_i^ν, $i \geq 1$, denote the time between the $(i-1)$th and ith birth. This means that T_i^ν is the time it takes for the population size to grow from i to $i+1$. More specifically, we will show that the sojourn times T_i^ν's are independent and T_i^ν is distributed

$$f_{T_i^\nu}(t) = i\lambda t^{\nu-1} E_{\nu,\nu}(-i\lambda t^\nu), \quad i \geq 1. \tag{4.1}$$

Recall that when $\nu = 1$, the inter-birth times T_i's of the Yp are independent and T_i is exponentially distributed with rate $i\lambda$, $i \geq 1$. Moreover, the waiting or birth time distribution for the pure linear birth process ($\nu = 1$) satisfies the following two equalities:

$$\Pr(\mathfrak{W}_j = T_1 + \cdots + T_j \leq t) = \Pr(\mathfrak{N}(t) \geq j+1 \big| \mathfrak{N}(0) = 1)$$

and

$$\mathfrak{p}_j(t) = \Pr(\mathfrak{W}_{j-1} \leq t) - \Pr(\mathfrak{W}_j \leq t).$$

Let $\mathfrak{W}_j^\nu = T_1^\nu + T_2^\nu + \cdots + T_j^\nu$ be the waiting time of the jth birth of the fYp. We now show that the preceding two equations hold true as well for the fractional or general case $(0 < \nu \leq 1)$, i.e.,

$$\Pr(\mathfrak{W}_j^\nu \leq t) = \Pr(\mathfrak{N}^\nu(t) \geq j+1 \big| \mathfrak{N}^\nu(0) = 1), \quad j \geq 1, \tag{4.2a}$$

and

$$\mathfrak{p}_j^\nu(t) = \Pr(\mathfrak{W}_{j-1}^\nu \leq t) - \Pr(\mathfrak{W}_j^\nu \leq t). \tag{4.2b}$$

Using (2.7), we obtain

$$\begin{aligned}
\Pr(\mathfrak{N}^\nu(t) \geq j+1 \big| \mathfrak{N}^\nu(0) = 1) &= \sum_{k=j+1}^{\infty} \Pr(\mathfrak{N}^\nu(t) = k \big| \mathfrak{N}^\nu(0) = 1) \qquad (4.3)\\
&= 1 - \sum_{k=1}^{j} \Pr(\mathfrak{N}^\nu(t) = k \big| \mathfrak{N}^\nu(0) = 1)\\
&= 1 - \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu).
\end{aligned}$$

This implies that the jth waiting time $\mathfrak{W}_j^\nu$ has distribution

$$f_{\mathfrak{W}_j^\nu}(t) = \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} (\lambda l) t^{\nu-1} E_{\nu,\nu}(-\lambda l t^\nu), \quad t > 0,\ \nu \in (0,1].$$

Integrating the preceding equation, we get

$$\int_0^\infty f_{\mathfrak{W}_j^\nu}(t)dt = \sum_{k=1}^{j}\sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1}$$

$$= \sum_{k=1}^{j} \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^l = \sum_{k=1}^{j} (1-1)^{k-1} = 1.$$

The non-negativity of $f_{\mathfrak{W}_j^\nu}(t)$ follows from the non-negativity of $\mathfrak{p}_k^\nu(t)$ (see Orsingher and Polito (2010)), and the last line of (4.3) is a monotone increasing function of t. To see this, we can write

$$\begin{aligned} & 1 - \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) \\ & = 1 - \sum_{k=1}^{j} \mathfrak{p}_k^\nu(t) \\ & = 1 - \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \Pr(T_l > t) \\ & = \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \Pr(T_l < t). \end{aligned}$$

Indeed, $f_{\mathfrak{W}_j^\nu}(t)$ is a probability density function. Note also that $f_{\mathfrak{W}_j^\nu}(t)$ has the following integral representation:

$$f_{\mathfrak{W}_j^\nu}(t) = \frac{1}{t} \int_0^\infty e^{-\xi} \sum_{k=1}^{j} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} g(l\lambda t/\xi) d\xi,$$

where $g(\eta) = \sin(\nu\pi)/[\pi(\eta^\nu + \eta^{-\nu} + 2\cos(\nu\pi))]$ (see Repin and Saichev (2000)). We now show that if the sojourn times are distributed as in (4.1), the cumulative distribution function $\Pr(\mathfrak{W}_j^\nu \le t)$ of the waiting or birth time equals the right-hand side of (4.2a). When $j = 1$, we get

$$\Pr(\mathfrak{W}_1^\nu \le t) = \Pr(T_1^\nu \le t) = 1 - E_{\nu,1}(-\lambda t^\nu) = 1 - \mathfrak{p}_1^\nu(t).$$

In the succeeding calculations, we use the following identities (see page 26 of Podlubny (1999)):

$$\int_0^t E_{\nu,1}(-j\lambda(t-u)^\nu) u^{\nu-1} E_{\nu,\nu}(-\lambda l u^\nu) du = \frac{j E_{\nu,\nu+1}(-j\lambda t^\nu) - l E_{\nu,\nu+1}(-l\lambda t^\nu)}{j-l} t^\nu$$

and

$$E_{\nu,\nu+1}(\xi) = \frac{E_{\nu,1}(\xi) - 1}{\xi}, \qquad l \le j.$$

Now,

$$\begin{aligned} \Pr(\mathfrak{W}_2^\nu \le t) & = \int_0^t \Pr\{T_1^\nu + T_2^\nu \le t \big| T_1^\nu = u\} dF_{T_1^\nu}(u) \\ & = \int_0^t [1 - E_{\nu,1}(-2\lambda(t-u)^\nu)] \lambda u^{\nu-1} E_{\nu,\nu}(-\lambda u^\nu) du \end{aligned}$$

$$\begin{aligned}
&= 1 - E_{\nu,1}(-\lambda t^\nu) - [2\lambda t^\nu E_{\nu,\nu+1}(-2\lambda t^\nu) - t^\nu E_{\nu,\nu+1}(-\lambda t^\nu)] \\
&= 1 - E_{\nu,1}(-\lambda t^\nu) - [E_{\nu,1}(-\lambda t^\nu) - E_{\nu,1}(-2\lambda t^\nu)] \\
&= 1 - 2E_{\nu,1}(-\lambda t^\nu) + E_{\nu,1}(-2\lambda t^\nu) \\
&= 1 - \sum_{k=1}^{2} \mathfrak{p}_k^\nu(t),
\end{aligned}$$

and in general, we can show by induction that

$$\begin{aligned}
&\Pr(\mathfrak{W}_j^\nu \le t) \\
&= \int_0^t \Pr\{\mathfrak{W}_j^\nu \le t \big| \mathfrak{W}_{j-1}^\nu = u\} dF_{\mathfrak{W}_{j-1}^\nu}(u) \\
&= \int_0^t [1 - E_{\nu,1}(-j\lambda(t-u)^\nu)] f_{\mathfrak{W}_{j-1}^\nu}(u) du \\
&= \int_0^t [1 - E_{\nu,1}(-j\lambda(t-u)^\nu)] \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} (\lambda l) u^{\nu-1} E_{\nu,\nu}(-\lambda l u^\nu) du \\
&= \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} [1 - E_{\nu,1}(-\lambda l t^\nu)] \\
&\quad - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \lambda l \int_0^t E_{\nu,1}(-j\lambda(t-u)^\nu) u^{\nu-1} E_{\nu,\nu}(-\lambda l u^\nu) du \\
&= \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} [1 - E_{\nu,1}(-\lambda l t^\nu)] \\
&\quad - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} [E_{\nu,1}(-\lambda l t^\nu) - E_{\nu,1}(-\lambda j t^\nu)] \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \left(\frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) - \frac{l}{j-l} E_{\nu,1}(-\lambda j t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad + \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda j t^\nu).
\end{aligned}$$

Using the formulas on page 3 of Gradshteyn and Ryzhik (1980), we have

$$\sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{l}{j-l} = \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \sum_{k=l}^{j-1} \binom{k-1}{l-1}$$

$$
\begin{aligned}
&= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \sum_{k=0}^{j-1-l} \binom{k+l-1}{l-1} \\
&= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \binom{j-1}{l} \\
&= \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} \frac{(j-1)!}{l!(j-l-1)!} \\
&= \sum_{l=1}^{j-1} \frac{(j-1)!}{(l-1)!(j-l)!} \\
&= \sum_{l=0}^{j-2} (-1)^l \binom{j-1}{l} = (-1)^{j-2},
\end{aligned}
$$

because

$$
\sum_{l=0}^{j-2} (-1)^l \binom{j-1}{l} = \sum_{l=0}^{j-1} (-1)^l \binom{j-1}{l} - (-1)^{j-1} \binom{j-1}{j-1}.
$$

Hence,

$$
\begin{aligned}
&\Pr(\mathfrak{W}_j^\nu \le t) \qquad (4.4) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) - (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} \binom{j-1}{l} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda l t^\nu) \right) \\
&= 1 - \sum_{k=1}^{j-1} \sum_{l=1}^{k} \binom{k-1}{l-1} (-1)^{l-1} \frac{j}{j-l} E_{\nu,1}(-\lambda l t^\nu) \\
&\quad - \left(\sum_{l=1}^{j} \binom{j-1}{l-1} (-1)^{l-1} E_{\nu,1}(-\lambda l t^\nu) - \sum_{l=1}^{j-1} (-1)^{l-1} \frac{l}{j-l} E_{\nu,1}(-\lambda l t^\nu) \sum_{k=l}^{j-1} \binom{k-1}{l-1} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \sum_{k=1}^{j-1}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1}\frac{j}{j-l}E_{\nu,1}(-\lambda l t^{\nu}) \\
&\quad - \left(\sum_{l=1}^{j}\binom{j-1}{l-1}(-1)^{l-1}E_{\nu,1}(-\lambda l t^{\nu}) - \sum_{l=k}^{j-1}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1}\frac{l}{j-l}E_{\nu,1}(-\lambda l t^{\nu})\right) \\
&= 1 - \sum_{k=1}^{j-1}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1}E_{\nu,1}(-\lambda l t^{\nu}) - \sum_{l=1}^{j}\binom{j-1}{l-1}(-1)^{l-1}E_{\nu,1}(-\lambda l t^{\nu}) \\
&= 1 - \sum_{k=1}^{j}\mathfrak{p}_k^{\nu}(t), \quad 1 \le k < j,
\end{aligned}
$$

as the second summation (in the preceding equal sign) simply corresponds to $k = j$. Hence, equality (4.2a) is attained. Again, the transition from the third equality to the fourth equality above uses formula (0.15.1) on page 3 of Gradshteyn and Ryzhik (1980), i.e.,

$$
\sum_{k=l}^{j-1}\binom{k-1}{l-1} = \sum_{k=0}^{j-l-1}\binom{k+l-1}{l-1} = \binom{j-1}{l}.
$$

Notice that when $\nu = 1$, we get $\Pr(\mathfrak{W}_j \le t) = (1 - e^{-\lambda t})^j$ which corresponds to the birth time distribution of the classical Yule process. Moreover, equality (4.2b) can be straightforwardly evaluated as

$$
\begin{aligned}
\Pr(\mathfrak{W}_{j-1}^{\nu} \le t) - \Pr(\mathfrak{W}_{j}^{\nu} \le t) &= \left(1 - \sum_{k=1}^{j-1}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1}E_{\nu,1}(-\lambda l t^{\nu})\right) \\
&\quad - \left(1 - \sum_{k=1}^{j}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1}E_{\nu,1}(-\lambda l t^{\nu})\right) \\
&= \sum_{l=1}^{j}\binom{j-1}{l-1}(-1)^{l-1}E_{\nu,1}(-\lambda l t^{\nu}) \\
&= \mathfrak{p}_j^{\nu}(t).
\end{aligned}
$$

In addition, the Laplace transform of the probability density $f_{T_i^{\nu}}(t)$ is

$$
\int_0^{\infty} e^{-zt} f_{T_i^{\nu}}(t)dt = \frac{i\lambda}{i\lambda + z^{\nu}}.
$$

This suggests that the distribution (eqn (4.1)) leads to the following known mixture or structural representation (see Cahoy et al. (2010)) of the inter-birth times as

$$
T_i^{\nu} \stackrel{d}{=} V_i^{1/\nu} S_{\nu},
$$

where V_i has the exponential distribution with parameter $i\lambda$, i.e.,

$$
f_{V_i}(v) = i\lambda e^{-i\lambda v}, \quad v > 0, \tag{4.5}
$$

and is independent of the positive *Lévy* or ν-stable distributed random variable S_ν having the Laplace transform of the density function e^{-z^ν}. This also suggests that the κ-*th* fractional moment of the ith inter-birth time is given by

$$\mathbf{E}\left[T_i^\nu\right]^\kappa = \frac{\pi\Gamma(1+\kappa)}{(i\lambda)^\kappa\Gamma(\kappa/\nu)\sin(\pi\kappa/\nu)\Gamma(1-\kappa)}, \quad 0<\kappa<\nu,$$

which further implies that the κ-*th* fractional moment of the jth wait or birth time is

$$\mathbf{E}\left[\mathfrak{W}_j^\nu\right]^\kappa = \frac{\pi\Gamma(1+\kappa)}{\lambda^\kappa\Gamma(\kappa/\nu)\sin(\pi\kappa/\nu)\Gamma(1-\kappa)}\sum_{k=1}^{j}\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{l-1}\left(\frac{1}{l^\kappa}\right),$$

where $0<\kappa<\nu$.

5 Sample paths of fYp

From Sections 3 and 4, it is now straightforward to simulate a trajectory of a fYp. However, we only propose the two simplest algorithms on how to generate a sample path of the fYp as the others follow. In particular, the random-rate representation (Representation A, Theorem 3.2) yields the algorithm below.

ALGORITHM 1:

i) Generate Ξ from the Wright distribution $W_{-\nu,1-\nu}(-\xi)$, and obtain ξ.
ii) Simulate a classical Yule process with birth rate $\lambda\xi$.
iii) Stretch the time scale to t^ν.

A simpler way to generate a realization of fYp with n births is to directly exploit the known birth and/or sojourn time distributions as follows: Generate V_i from the exponential distribution in (4.5) with parameter $i\lambda$, and S_ν from the strictly positive stable distribution with parameter ν.

ALGORITHM 2:

i) Let $i=1$ and $\mathfrak{N}^\nu(0)=1$.
ii) Simulate $T_i^\nu = V_i^{1/\nu}S_\nu$, and let $\mathfrak{W}_i^\nu = T_1^\nu + T_2^\nu + \cdots + T_i^\nu$.
iii) $\mathfrak{N}^\nu(\mathfrak{W}_i^\nu) = i+1$, and $i=i+1$.
iv) Repeat ii-iii for $i=2,\ldots,n-1$.

We now use the algorithms above to highlight some unique properties of the fractional Yule process that are related to its true mean given in (2.3). Figure 5.1 below shows both Yp and fYp as jump processes of size 1 in the time interval $(0,5)$ with $\nu=0.5$, and $\lambda=1$. Using the same set of parameters, Figure 5.2 displays sample trajectories of a different/independent fYp and Yp which model a binary-split growth process. An important attribute that can be directly observed from these two graphs is that on the average, fYp grows more rapidly than the classical Yp shortly after it starts.

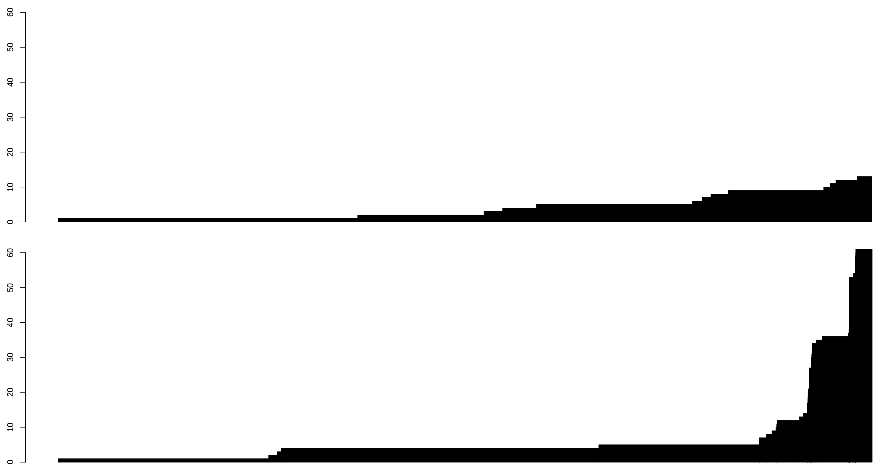

Fig. 5.1 *Sample trajectories of the standard Yule process (top) and the fractional Yule process (bottom) in the interval* $(0, 5)$ *with parameters* $(\nu, \lambda) = (0.5, 1)$.

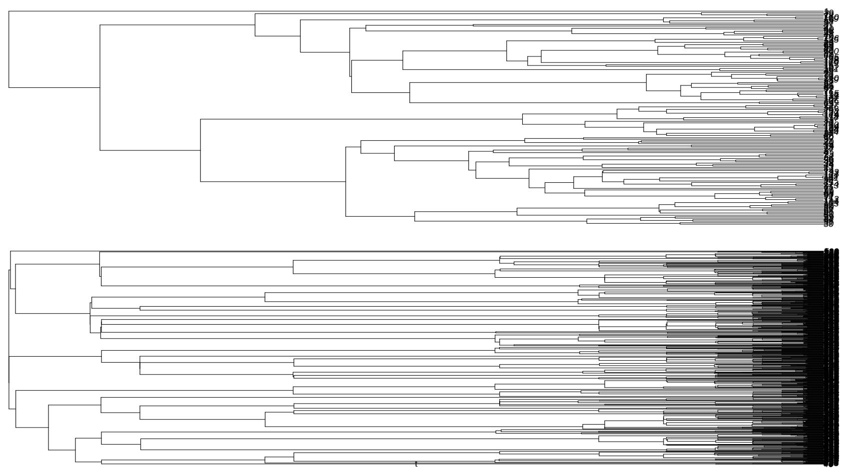

Fig. 5.2 *Sample paths of the pure linear birth process (top) and the fractional Yule process (bottom) in the interval* $(0, 5)$ *with parameters* $(\nu, \lambda) = (0.5, 1)$.

In addition, a more specific characteristic of fYp is illustrated in Figure 5.3. The particular realization of fYp below used the parameter values $\nu = 0.25$, $\lambda = 1$, and is observed in the time interval $(0, 5)$. It clearly suggests that fYp is more explosive than Yp when $\nu \to 0$. In general, the plots strongly validate the plausibility of fYp to model exploding and strictly growing processes. Note also that Representation A implies that the interaction between the random rate and time stretching of the classical Yule process can rapidly speed up or slow down fYp at any given time instance.

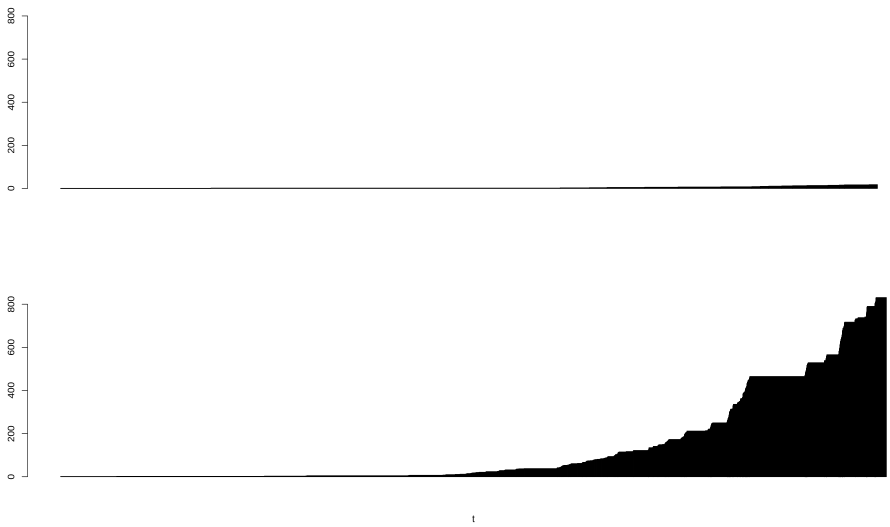

Fig. 5.3 *Sample paths of the classical Yule process (top) and the fractional Yule process (bottom) in the interval* $(0,5)$ *with parameters* $(\nu, \lambda) = (0.25, 1)$.

6 Method-of-Moments (MoM) estimation

We now propose a method-of-moments estimation procedure for the parameters ν and λ to make fYp usable in practice. In this procedure, we assume that a particular realization or complete history of the process is observed until the population is n, i.e., there are n births. We then attempt to use all the available data from the observed sample path of the fractional Yule process.

In particular, we use all the available inter-birth or sojourn times of the observed sample trajectory of the fractional Yule process. A direct way of estimating the parameters is to use the fractional moment estimators as follows: Choose constants $\kappa_m < \nu, m = 1,2$, and solve for the estimates $\hat{\lambda}$ and $\hat{\nu}$ using the equations

$$\frac{\sum_{i=1}^n [T_i^\nu]^{\kappa_m}}{n} = \frac{\pi \Gamma(1+\kappa_m)}{\hat{\lambda}^{\kappa_m} \Gamma(\kappa_m/\hat{\nu}) \sin(\pi\kappa_m/\hat{\nu}) \Gamma(1-\kappa_m)} \frac{\sum_{i=1}^n 1/i^{\kappa_m}}{n}, \quad m = 1,2.$$

Another approach is to use the first two integer-order moments of the log-transformed sojourn times (see Cahoy et al. (2010)) which are

$$\mathbf{E} \ln [T_i^\nu] = \frac{-\ln(i\lambda)}{\nu} - \gamma,$$

and

$$\mathbf{E} \ln [T_i^\nu]^2 = \pi^2 \left(\frac{1}{3\nu^2} - \frac{1}{6} \right) + \left(\frac{\ln(i\lambda)}{\nu} + \gamma \right)^2.$$

This further suggests that the parameter estimates can be computed using the two equations:

$$\frac{\sum_{i=1}^{n} \ln\left[T_i^{\nu}\right]}{n} = \frac{-\sum_{i=1}^{n} \ln(i\lambda)}{\nu n} - \gamma,$$

and

$$\frac{\sum_{i=1}^{n} \left(\ln\left[T_i^{\nu}\right]\right)^2}{n} = \pi^2 \left(\frac{1}{3\nu^2} - \frac{1}{6}\right) + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\ln(i\lambda)}{\nu} + \gamma\right)^2,$$

where $\gamma \cong 0.577215664901532$ is the Euler–Mascheroni constant. A major advantage of this procedure over other moment estimators is that it does not require selection of constants a priori to calculate the parameter estimates. Note also that the maximum likelihood estimators are more challenging to compute due to the required evaluation of the Mittag–Leffler function.

In addition, we tested our parameter estimation procedure. In doing so, we generated 10 random samples of inter-birth times of size 10000 each for $\nu = 0.1 + 0.1m$, $m = 0, \ldots, 9$ and $\lambda = 0.2, 10$. For each simulated data set, we computed the estimates using the first n observations in the set with $n = 100, 1000$, and 10000. The tables below show the simulation results for a single run, which further indicate that the proposed procedure performs relatively well as the sample sizes increase. Please note that in many applications (e.g., internet traffic), the typical number of observations is at least of the order of millions. These estimates could also serve as good starting values of an iterative estimation procedure.

Table 6.1 *Parameter estimates* $(\widehat{\nu}, \widehat{\lambda})$ *for fYp with* $\nu = 0.1(0.1)1$ *and* $\lambda = 0.2$.

	$n = 100$	$n = 1000$	$n = 10000$
$(\nu = 0.1, \lambda = 0.2)$	(0.095, 0.198)	(0.096, 0.185)	(0.100, 0.205)
$(\nu = 0.2, \lambda = 0.2)$	(0.228, 0.249)	(0.193, 0.189)	(0.199, 0.193)
$(\nu = 0.3, \lambda = 0.2)$	(0.283, 0.185)	(0.292, 0.193)	(0.303, 0.228)
$(\nu = 0.4, \lambda = 0.2)$	(0.381, 0.178)	(0.407, 0.218)	(0.402, 0.209)
$(\nu = 0.5, \lambda = 0.2)$	(0.481, 0.212)	(0.501, 0.197)	(0.500, 0.197)
$(\nu = 0.6, \lambda = 0.2)$	(0.599, 0.211)	(0.602, 0.186)	(0.595, 0.186)
$(\nu = 0.7, \lambda = 0.2)$	(0.759, 0.257)	(0.728, 0.250)	(0.700, 0.198)
$(\nu = 0.8, \lambda = 0.2)$	(0.818, 0.220)	(0.819, 0.229)	(0.803, 0.204)
$(\nu = 0.9, \lambda = 0.2)$	(0.850, 0.193)	(0.899, 0.211)	(0.907, 0.215)
$(\nu = 1.0, \lambda = 0.2)$	(0.977, 0.183)	(0.991, 0.199)	(0.999, 0.202)

7 Concluding remarks

We have derived one-dimensional representations of the fractional Yule process, which led to algorithms for simulating its sample paths. These representations are also necessary in understanding the properties of fYp further. We have derived the birth and inter-birth or sojourn time distributions, which are of Mittag–Leffler type. The structural representation of the random sojourn time also led to an algorithm for simulating sample trajectories of the

Table 6.2 *Parameter estimates* $(\widehat{\nu}, \widehat{\lambda})$ *for fYp with* $\nu = 0.1(0.1)1$ *and* $\lambda = 10$.

	$n = 100$	$n = 1000$	$n = 10000$
$(\nu = 0.1, \lambda = 10)$	(0.107, 13.067)	(0.101, 10.599)	(0.101, 10.730)
$(\nu = 0.2, \lambda = 10)$	(0.203, 10.737)	(0.206, 12.384)	(0.201, 10.555)
$(\nu = 0.3, \lambda = 10)$	(0.299, 11.027)	(0.297, 9.359)	(0.295, 8.593)
$(\nu = 0.4, \lambda = 10)$	(0.391, 7.598)	(0.396, 8.899)	(0.397, 9.086)
$(\nu = 0.5, \lambda = 10)$	(0.517, 10.939)	(0.509, 11.428)	(0.501, 10.269)
$(\nu = 0.6, \lambda = 10)$	(0.630, 11.379)	(0.586, 8.308)	(0.597, 9.162)
$(\nu = 0.7, \lambda = 10)$	(0.716, 12.413)	(0.699, 10.634)	(0.710, 11.679)
$(\nu = 0.8, \lambda = 10)$	(0.782, 8.713)	(0.786, 8.186)	(0.804, 10.498)
$(\nu = 0.9, \lambda = 10)$	(0.919, 11.429)	(0.899, 9.043)	(0.897, 9.684)
$(\nu = 1.0, \lambda = 10)$	(0.969, 8.712)	(1.000, 10.427)	(1.001, 10.434)

fYp. We have proposed an estimation procedure using the moments of the log-transformed inter-birth times, which performed satisfactorily especially for larger sample sizes.

Although some properties of fYp have already been studied, there are still a lot of open problems that need to be figured out. For instance, understanding fYp in more depth and the construction of more efficient estimators like the maximum likelihood would be worth pursuing in the future. Also, the application of fYp in practice particularly in biology and/or network traffic is still in progress.

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